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THE DYNAMICS OF PYRAMIDAL BODIES WITHIN THE FRAMEWORK OF THE LOCAL INTERACTION MODEL†

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Two-dimensional inertial motion of pyramidal bodies in a medium is investigated, on the assumption that the force exerted by the medium on their surface is described by the local interaction model. Assuming unseparated flow around the bodies and small perturbations applied at the initial time to the parameters of rectilinear motion, an analytical solution is constructed of the problem of the two-dimensional motion of slender bodies with bases whose contour is a rhombus or a star consisting of four symmetrical cycles. It is shown that the solution provides the basis for a complete parametric analysis of the dynamics of the body and for evaluating the forces and torques experienced by the body along its trajectory. A criterion for the stability of the body is found, using which, knowing the velocity, mass and position of the body's centre of gravity, one can determine the form of the perturbed motion of the pyramidal body. It is shown that the body shape is one of the most important factors affecting the stability of motion, and that, of all bodies with the same shape and position of the centre of mass, those with the least mass have the largest reserve of stability. The analytical results are confirmed by numerical solution of the Cauchy problem for the system of equations of motion obtained without the simplifying assumptions.

Problems relating to optimizing the shape of a body (for drag and depth of penetration) have been solved using the local interaction model (LIM) [1–3], on the assumption that the base area and length of the body are given, without simplifying assumptions about the body geometry [4–6]. It has been shown that bodies of minimum drag [4–6] and maximum depth of penetration [7] are formed by pieces of surfaces whose normal makes the optimum angle with the direction of motion, where the optimum angle is determined by the characteristics of the medium and the initial velocity of motion. It has been shown [7] that in the general case bodies of maximum depth of penetration and bodies of minimum drag have different optimum angles. A method has been proposed for constructing optimum shapes, using which one can design an infinite set of optimum configurations, including pyramidal configurations consisting of pieces of planes tangent to the optimum cone [4–6].

Optimum shapes have been constructed for rectilinear motion of the body [4–7]. The motion of the body in a real medium may be perturbed, and then, as investigations of the dynamics of slender axisymmetric bodies have shown [8–10], the velocity of the centre of mass may deviate considerably from its initial direction, and the body's trajectory of motion is strongly curved. Intensifying the perturbations may cause the body to overturn, and then the theoretically predicted depth of penetration is unattainable. Thus, the bodies constructed in [4–7] will possess the promised optimum characteristics only if their rectilinear motion is stable to small perturbations of the initial parameter values.

Investigation of the effect of perturbations on the characteristics of the motion of the body and the development of stability criteria are important stages in research on the properties of optimum bodies. A complete solution of the problem of the dynamics of the body, including the construction of the trajectory of motion, will enable an exact solution of the problem to be obtained. However, the problem is multiparametric even for two-dimensional motion of slender bodies of revolution, and an analytical solution has been found only for a slender cone and for motion in which the medium flows around the body without separation [8, 9]. The presence of zones in which the medium separates at the body surface complicates the problem, and in that case analytical investigation of the problem has been confined to an analysis of the stability of the motion of the body at the initial stage [10]. Numerical investigations of the problem of the dynamics of the body have shown that a solution within the framework of the LIM may be found for any initial conditions and with no restrictions on the body's mode of motion [3]. However, a numerical solution, while of undoubted value, cannot lead to the general laws of motion.

In what follows, the LIM will be used to carry out a numerical and analytical investigation of the

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two-dimensional inertial motion of pyramidal bodies formed, like the optimum bodies of [4–7], by pieces of planes tangent to a circular cone.

1. THE MODEL OF THE INTERACTION OF THE MEDIUM AND THE BODY

Consider the inertial motion of a body in a medium, on the assumption that the body is completely immersed in the medium at the starting time and that the influence of the medium's free surface on the motion of the body is negligible.

We will write the force exerted by the medium on the body in the form

$$\mathbf{F} = \iint_S [\sigma_n \mathbf{n} + \sigma_\tau \boldsymbol{\tau}] dS \qquad (1.1)$$

where σ_n and σ_τ are the normal and shear stresses at the body surface, $\mathbf{n}$ and $\boldsymbol{\tau}$ are the unit vectors of the inner normal and the tangent to the surface element, and integration is carried out over the body-medium contact surface S.

Let us assume that each element of the surface S interacts with the medium independently of other parts of the body, and that the force exerted on it can be described by a local model. To denote the normal stress we shall use a two-term formula with a dynamic term and a strength term; the friction at the body surface is assumed to be constant:

$$\sigma_n = A_1(\mathbf{U} \cdot \mathbf{n})^2 + C_1, \quad \sigma_\tau = C_2 \qquad (1.2)$$

The positive coefficients A_1, C_1 and C_2 are constant parameters of the model, determined by the characteristics of the medium, $\mathbf{U}$ is the total velocity of an element of the surface: $\mathbf{U} = \mathbf{U}_c + [\boldsymbol{\Omega} \times \mathbf{r}]$, where $\mathbf{U}_c$ is the velocity of the centre of mass, $\boldsymbol{\Omega}$ is the angular velocity of revolution of the body, and $\mathbf{r}$ the radius vector of the element with its initial point at the centre of mass.

Model (1.2) is a special case of the representation of the stresses within the framework of the LIM. It has been shown [11] that, with certain assumptions, formulae (1.2) describe the stresses at the surface of the body as it moves in a gas or in dense media like soils or metals. The term C_1 in that case characterizes the resistivity of the medium to deformation, and the coefficient A_1 is of the order of magnitude of the density of the medium. The constant friction model (1.2) is frequently used to calculate the forces acting on a body penetrating into elastoplastic media [10], in which case $C_2 = \tau_s$, where τ_s is the plastic friction. The values of A_1, C_1 and C_2 for actual media are either obtained by solving model problems [1–3] or determined experimentally. For example, for argillaceous media, according to the solution obtained for an incompressible elastoplastic medium [3], one can take

$$A_1 = 3\rho_0/2, \quad C_1 = 4\tau_s(1 + \ln(\mu/\tau_s))/3, \quad C_2 = \tau_s \qquad (1.3)$$

where ρ_0 is the density of the medium and μ is the shear modulus.

Within the scope of the LIM, the vector τ is collinear with the vectors $\mathbf{U}$ and $\mathbf{n}$:

$$\boldsymbol{\tau} = [[\mathbf{U} \times \mathbf{n}] \times \mathbf{n}]/|[\mathbf{U} \times \mathbf{n}]| \qquad (1.4)$$

and the surface S is defined by the condition

$$(\mathbf{U} \cdot \mathbf{n}) \le 0 \qquad (1.5)$$

The selected model enables us to express the force $\mathbf{F}$ of (1.1) explicitly as a function of the body shape and the parameters of the motion, in which the characteristics of the medium occur as constants.

2. THE CLASS OF CONFIGURATIONS UNDER INVESTIGATION

The body shape is assumed to be given and consists of pieces of planes tangent to a circular cone.

In a cylindrical system of coordinates (r, θ, x_1) with origin O_1 at the vertex of the body and axis O_1x_1 pointing along the axis of the cone, the equation of the surface of the cone is

$$r = r(\theta, x_1) = r_1 x_1$$

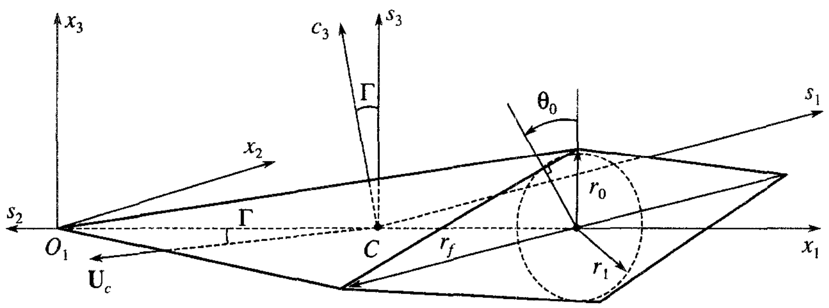

Fig. 1

where $r_1 = \mathrm{tg}\alpha_1$, α_1 being the semi-vertex angle of the cone. We shall assume that α_1, as well as the length **L** and the base area S_b, are given and all linear dimensions are in units of L: $x_1 \in [0, 1]$.

The class of configurations to be investigated will be limited to pyramidal bodies with two and four planes of symmetry, the shape of whose base is a rhombus or a symmetric four-pointed star whose sides belong to straight lines tangent to the circle of radius r_1.

An example of a rhomboid configuration is shown in Fig. 1, where the $O_1x_1x_2$ and $O_1x_1x_3$ planes of the Cartesian system of coordinates $O_1x_1x_2x_3$ are the planes of symmetry of the body. The contour of the base is a rhombus and the radii of its vertices are related to r_1 as follows:

$$r_0 = r_1/\cos\theta_0, \quad r_f = r_1/\sin\theta_0 \tag{2.1}$$

The angle θ_0 is determined from S_b by the condition

$$2r_0r_f = \pi r_b^2, \quad r_b = (S_b/\pi)^{1/2}/L$$

Setting

$$b = (R_1/R_k)^2; \quad R_1 = r_1/r_b, \quad R_k = \pi^{1/2}/2 \approx 0.89$$

we write θ_0 in the form

$$\theta_0 = \arccos((P/2)^{1/2}), \quad P = P_{1,2} = 1 \pm (1-b^2)^{1/2} \tag{2.2}$$

For $b < 1$ ($R_1 < R_k$) there are two values of P: $P_1 \in [1, 2)$ and $P_2 \in (0, 1]$ and two angles θ_0. To each of these corresponds a rhomboid body whose surface is described by the following equation in (r, θ, x_1)

$$r = r(\theta, x_1) = \frac{\xi_1 r_1 x_1}{\cos(\theta - \xi_1\xi_2\theta_0)}; \quad \xi_1 = \mathrm{sign}(\cos\theta), \quad \xi_2 = \mathrm{sign}(\sin\theta) \tag{2.3}$$

The angle θ is measured from the O_1x_3 axis.

If $P = P_1$, then $\theta_0 < \pi/4$, $r_0 < r_f$, and r_f is the maximum radius of the body (see Fig. 1). If $P = P_2$, then $\theta_0 > \pi/4$, $r_0 > r_f$, and the maximum radius of the body is r_0. We will call a configuration with $r_0 < r_f$ horizontal, and one with $r_0 > r_f$ vertical. Each of these shapes is obtained from the other by a 90° rotation about the O_1x_1 axis.

An example of a pyramidal body with star-shaped base contour is shown in Fig. 2, from which it follows that a star-shaped body is formed by pieces of the surfaces of two rhomboid bodies (2.3), rotated through 90° with respect to each other. Since the body is constructed for a given base area S_b, the angles θ_0 for the vertical and horizontal configurations (2.3) are equal to θ_1 and θ_2, respectively (see Fig. 2b), where

$$\theta_1 = \mathrm{arctg}([1 + [1 + 4b(1-b)]^{1/2}]/(2b)), \quad \theta_2 = \pi/2 - \theta_1 \tag{2.4}$$

The constructions of bodies just described are possible for $R_1 \leqslant R_k$ when $b \leqslant 1$. If $R_1 = R_k$, then $b = 1$ and, as follows from (2.2) and (2.4), $\theta_0 = \pi/4$ for all shapes. Then the base contour of the body is a square and all the bodies have the same geometry.

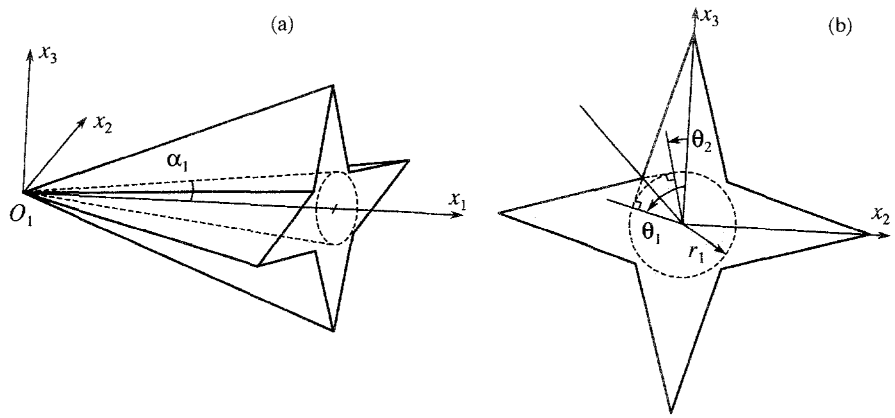

Fig. 2

3. THE EQUATIONS AND PARAMETERS OF MOTION

Let us assume that each point of the body describes a trajectory in a plane parallel to the plane of symmetry of the body, $O_1x_1x_3$. Then the vector Ω of the angular velocity of revolution of the body is normal to the $O_1x_1x_3$ plane and, relative to the system of coordinates $O_1x_1x_2x_3$, has a single non-zero component Ω.

Suppose the centre of mass of the body is at a point C on the O_1x_1 axis, at a distance C_m from the vertex of the body: $C_m \in (0, 1)$. We introduce two right-handed systems of coordinates: a stationary system $OXYZ$, whose origin coincides with the point C at the starting time, and a moving system of coordinates $Cs_1s_2s_3$ rigidly attached to the body.

We shall assume that the OX axis is pointing along the O_1x_2 axis, while the OY axis is taken to correspond with the direction of the gravity force, making an angle φ_0 with the velocity vector $\mathbf{U}_c$ of the centre of mass at the starting time. Then the vector $\mathbf{U}_c$ always lies in the OYZ plane and the position of the centre of mass relative to the system $OXYZ$ at time t is defined by two coordinates, $Y(t)$ and $Z(t)$:

$$\frac{dY}{dt} = \frac{U_c}{L}\cos\varphi, \quad \frac{dZ}{dt} = \frac{U_c}{L}\sin\varphi \tag{3.1}$$

where $U_c = |\mathbf{U}_c|$, and $\varphi = \varphi(t)$ is the angle between $\mathbf{U}_c$ and the OY axis, where $\varphi(0) = \varphi_0$.

The axes of the moving system of coordinates $Cs_1s_2s_3$ are attached to the body, being placed in the planes of symmetry of the body so that $\mathbf{s}_1 = \mathbf{x}_2$, $\mathbf{s}_2 = -\mathbf{x}_1$ and $\mathbf{s}_3 = \mathbf{x}_3$, where $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are the unit vectors of the system of coordinates $O_1x_1x_2x_3$ (see Fig. 1). The direction of the vector $\mathbf{U}_c$ in the system of coordinates $Cs_1s_2s_3$ is defined by the angle Γ: $\mathbf{U}_c = U_c\{0, \cos\Gamma, -\sin\Gamma\}$.

In two-dimensional motion, with a known distribution of the forces exerted on the body, the quantities $U_c(t)$, $\varphi(t)$, $\Omega(t)$ and $\Gamma(t)$ are found as functions of t from the dynamic equations of motion (see, e.g. [12]), which can be written as follows for a body of mass m and moment of inertia I_1 about the Cs_1 axis:

$$\frac{dU_c}{dt} = \frac{F_2}{m}, \quad \frac{d\varphi}{dt} = \frac{F_3}{mU_c}, \quad \frac{d\Omega}{dt} = \frac{M_1}{I_1}, \quad \frac{d\Gamma}{dt} = \Omega - \frac{F_3}{mU_c} \tag{3.2}$$

where M_1 is the component of the moment of the forces about the s_1 axis, and F_2 and F_3 are the components of the resultant force in the respective directions of the vector $\mathbf{U}_c$ and a vector $\mathbf{c}_3$ normal to $\mathbf{U}_c$, whose form in the attached system of coordinates is $\mathbf{c}_3 = \{0, \sin\Gamma, \cos\Gamma\}$ (see Fig. 1). The forces acting on the body, which is moving by inertia through the medium, are the forces due to its interaction with the medium and also the gravity force; hence

$$F_2 = F_{c2} + mg\cos\varphi, \quad F_3 = F_{c3} - mg\sin\varphi$$

where g is the acceleration due to gravity, and F_{c2} and F_{c3} are the components of the force $\mathbf{F}$ of (2.2), derived using model (1.2).

For the surface (2.3), the vectors $\mathbf{n}$ and $\mathbf{U}$ occurring in the first formula of (1.2) and the scalar product $(\mathbf{U} \cdot \mathbf{n})$ may be expressed as follows:

$$\begin{aligned}
\mathbf{n} &= (1+\alpha^2)^{-1/2}\{\xi_2\sin\theta_0, -\alpha, -\xi_1\cos\theta_0\}\\
\mathbf{U} &= U_c\cos\Gamma\{0, 1-\omega r_b r\cos\theta, -r_b(\gamma-\omega(C_m-x_1))\}\\
(\mathbf{U}\cdot\mathbf{n}) &= r_1 U_c\cos\Gamma(-1+\omega r_b r\cos\theta+\xi_1(\gamma-\omega(C_m-x_1))/(R_0\cos\alpha_1))
\end{aligned} \tag{3.3}$$

where

$$\alpha = (\mathbf{n}\cdot\mathbf{x}_1) = \sin\alpha_1, \quad R_0 = r_0/r_b, \quad \gamma = \mathrm{tg}\Gamma/r_b, \quad \omega = \Omega L/(r_b U_c\cos\Gamma)$$

(γ and ω are non-dimensional parameters of the motion of the body about its centre of mass). Here and below, the coordinates of the vectors are given in the attached system of coordinates.

Substituting formulae (3.3) into Eqs (1.2) and (1.4) and integrating in (1.1) over the surface S (1.5), one can express the force $\mathbf{F}$ and its components as functions of the variables U_c, Γ and Ω. The system of equations (3.2) may be integrated if the initial values of U_c, φ, Ω and Γ are known. However, if it is assumed that the medium may separate from the lateral surface of the body, the surface S will be determined during the solution process, and in that case the system can be integrated only numerically.

A program has been written to compute the dynamics of the pyramidal body (2.3); it is based on solving a Cauchy problem for the system of equations of motion (3.1) and (3.2), without restrictions on the form of motion of the body. Computations using this program will be used below in Section 7 to confirm the analytical results obtained in Sections 4–6 under simplifying assumptions.

4. BASIC ASSUMPTIONS OF THE THEORY

An analytical solution of the system of equations (3.1) and (3.2) will be constructed for slender bodies ($r_b^2 \ll 1$), on the assumption that the medium does not separate from the lateral surface of the body. Then the surface of integration S in (1.1) is the entire surface of the body (2.3). Condition (1.5) must hold on that surface, from which, taking (3.3) into consideration, we obtain the following restrictions on γ and ω

$$|\gamma-\omega C_m| \le R_0, \quad |\gamma+\omega(1+r_0^2-C_m)| \le R_0 \tag{4.1}$$

The quantity R_0 for star-shaped bodies is found from r_0 (2.1) with $\theta_0 = \theta_2$ (2.4). Hence it follows that $|\gamma| \le R_0$, $|\omega| \le 2R_0$, and if we assume in addition that

$$r_0^2 \ll 1 \tag{4.2}$$

then formulae (3.3) and the vector τ (1.4) can be written as

$$\begin{aligned}
\mathbf{n} &= \{\xi_2\sin\theta_0, -\alpha, -\xi_1\cos\theta_0\}\\
\mathbf{U} &= U_c\{0, 1, -\beta(\gamma-\omega(C_m-x_1))\}\\
(\mathbf{U}\cdot\mathbf{n}) &= \alpha U_c[-1+\xi_1(\gamma-\omega(C_m-x_1))/R_0]\\
\boldsymbol{\tau} &= -\mathbf{u}-\alpha[1-\xi_1(\gamma-\omega(C_m-x_1))/R_0]\mathbf{n}, \quad \mathbf{u} = \mathbf{U}/U_c
\end{aligned} \tag{4.3}$$

where

$$\gamma = \Gamma/\beta, \quad \omega = \Omega L/(\beta U_c)$$

β being the semi-vertex angle of a circular cone of length L with base area S_b: $\beta = \mathrm{arctg}\, r_b \approx r_b$ ($\beta^2 \ll 1$)

Taking the above assumptions into consideration, and letting the length of the trajectory of the centre of mass l: $dl = (U_c/L)dt$ be the independent variable, we can rewrite Eqs (3.2) in terms of the variables U_c, φ, γ and ω

$$\frac{dU_c}{dl} = f_2 U_c, \quad \frac{d\varphi}{dl} = f_3, \quad \frac{d\gamma}{dl} = \omega-\frac{f_3}{\beta}, \quad \frac{d\omega}{dl} = \frac{m_1}{i_1\beta}-\omega f_2; \quad i_1 = \frac{I_1}{mL^2} \tag{4.4}$$

The non-dimensional quantities m_1, f_2 and f_3 are defined in terms of the components M_1, F_2 and F_3

$$m_1 = M_1/N, \quad f_2 = F_2L/N, \quad f_3 = F_3L/N; \quad N = mU_c^2$$

Ignoring the gravity force and using formulae (4.1)–(4.3) to evaluate the moment and components of $\mathbf{F}$ (1.1), we can write m_1, f_2 and f_3 in the form

$$\begin{aligned} m_1 &= -A_mP_f\beta[\gamma z_f + \omega(z_cz_f + E_2/18)] \\ f_2 &= -A_m\alpha^2(1 + D + [\gamma(\gamma + \omega z_c) + (\gamma + \omega z_c)^2/2 + \omega^2/36]P_f/R_1^2) \\ f_3 &= A_mP_f\beta(\gamma E_1 + \omega z_cE_2) \end{aligned} \tag{4.5}$$

We have used the following notation

$$\begin{aligned} A_m &= 3A_1/\rho_m, \quad z_c = \frac{2}{3} - C_m, \quad z_f = z_cE_2 + 2\alpha^2/(3P_f) \\ D &= D_1 + D_2, \quad D_1 = C_1/(A_1U_c^2\alpha^2), \quad D_2 = C_2/(A_1U_c^2\alpha^3) \\ E_1 &= 1 - \alpha^2(1 + D_1 + D_2P_f/2)/P_f, \quad E_2 = 1 + \alpha^2D_2(2 - P_f)/(2P_f) \end{aligned} \tag{4.6}$$

where ρ_m is the average density of the body: $\rho_m = m/V$, V is the volume of the body and P_f is the shape parameter. For star-shaped bodies, $P_f = 1$, and for rhomboid bodies $P_f = P$, where P is found from the second formula of (2.2).

In rectilinear motion, $\varphi = \varphi_0$, $\gamma = \omega = 0$, and the first equation of (4.4) can be written in terms of D in the form

$$dD/dl = 2A_m\alpha^2D(1 + D) \tag{4.7}$$

Its solution yields the relation of D and U_c with the length of the trajectory of the centre of mass

$$\begin{aligned} l &= l_m - \ln(1 + 1/D)/(2A_m\alpha^2) \\ l_m &= \ln(1 + 1/D_0)/(2A_m\alpha^2) \\ D_0 &= D(U_0) = (C_1 + C_2/\alpha)/(A_1U_0^2\alpha^2) \end{aligned} \tag{4.8}$$

where l_m is the total path length and U_0 is the initial velocity of motion of the body.

If the body is moving in the medium along the normal to its free surface, l_m is the depth of penetration of the body. It has been shown [7] that, if the initial velocity U_0 is known, the maximum of l_m for all bodies of given mass, length and base area is achieved for bodies constructed of pieces of surfaces whose normals make the optimum angle with the direction of motion. This angle may be found by considering l_m as a function of α, $\alpha \in [0, 1]$, and finding $\alpha = \alpha^*$ for which $l_m(\alpha)$ is a maximum. The number α^* is independent of the mass, length and base area of the body; it is determined by the initial velocity U_0 and constant parameters of the model (1.2).

For known U_0 and α^*, the pyramidal bodies (2.3) are bodies with maximum depth of penetration if $r_1 = \alpha^*$. However, they can only be used effectively when their rectilinear motion is stable to small perturbations of the initial values of the parameters of motion of the body about its centre of mass.

5. MOTION OF THE BODY ABOUT THE CENTRE OF MASS

Stability criterion. The stability of rectilinear motion of the body will be analysed, and a solution of the system of equations (4.4) found, on the additional assumption that the perturbations applied to γ and ω are small and satisfy the condition

$$(\gamma^2 + \omega^2)P_f/R_1^2 \ll 1 \tag{5.1}$$

In that case the terms involving γ and ω may be eliminated from the expression for f_2 and then, as in the case of rectilinear motion, the velocity of the centre of mass U_c will be found from formulae (4.8).

Using formulae (4.5) and notation (4.6), we can rewrite Eqs (4.4) for γ and ω as

$$\frac{d\gamma}{dl} = -(1 - A_m P_f z_c E_2)(K_\gamma E_1 \gamma - \omega), \quad \frac{d\omega}{dl} = A_m P_f \chi (K_\omega \gamma - \omega) \tag{5.2}$$

where

$$\chi = \frac{1}{i_1}\left[z_c z_f + \frac{E_2}{18} - i_1(1 + D)\frac{\alpha^2}{P_f}\right], \quad K_\gamma = \frac{A_m P_f}{1 - A_m P_f z_c E_2}, \quad K_\omega = -\frac{z_f}{i_1 \chi} \tag{5.3}$$

Rectilinear motion of the body is stable if small perturbations of γ and ω decay in time. This will be the case if the trivial solution of system (5.2) is asymptotically stable in Lyapunov's sense (see, e.g. [13]) and all the roots of the characteristic equation of the system have negative real parts.

Considering D, D_1, and D_2 as constant parameters, let us write the characteristic equation and conditions for asymptotic stability of the trivial solution of system (5.2) as follows:

$$\begin{gathered} \lambda^2 + b_1\lambda + b_2 = 0 \\ b_1 = A_m P_f(\chi + E_1) > 0, \quad b_2 = \chi(A_m P_f)^2(E_1 - K_\omega/K_\gamma) > 0 \end{gathered} \tag{5.4}$$

Inequalities (5.4) determine the stability of the trivial solution of system (5.2) in the case when its coefficients are constant. The dependence of D on l violates this condition, but if we consider values of D less than or of the order of unity, then

$$D\alpha^2/P_f \ll 1 \tag{5.5}$$

and the parameters D, D_1 and D_2 may be eliminated from the expressions for the coefficients.

Then, taking into account that $i_1 \sim 10^{-1}$, we deduce from formulae (4.6) and (5.3) that

$$z_f = z_c + 2\alpha^2/(3P_f), \quad \chi = (z_c z_f + 1/18)/i_1, \quad E_1 = E_2 = 1 \tag{5.6}$$

and the system of equations (5.2) becomes autonomous. In that case its solution is independent of the parameters of motion of the centre of mass, and the conditions for the stability of its trivial solution (5.4) may be written as a single inequality

$$K_\omega/K_\gamma < 1 \tag{5.7}$$

We will construct a solution of system (5.2) observing condition (5.5) in parametric form, using the variable $\eta = \omega/\gamma$. The equation for η, in accordance with Eqs (5.2) and relations (5.6), may be written as

$$d\eta/dl = -(1 - A_m P_f z_c)\Phi(\eta), \quad \Phi(\eta) = \eta^2 + K_\gamma(\chi - 1)\eta - \chi K_\omega K_\gamma$$

If $\Lambda \neq 0$, where Λ is the discriminant of the characteristic equation (5.4)

$$\Lambda = (A_m P_f)^2 \Delta, \quad \Delta = (\chi + 1)^2 - 4\chi(1 - K_\omega/K_\gamma)$$

the quadratic equations (5.4) and $\Phi(\eta) = 0$ have two roots each:

$$\lambda_{1,2} = A_m P_f(-(\chi + 1) \pm \Delta^{1/2})/2, \quad \eta_{1,2} = K_\gamma(1 - \chi \pm \Delta^{1/2})/2 \tag{5.8}$$

We can thus write the general solution of system (5.2) as

$$\gamma = h_1\exp(\lambda_1 l) + h_2\exp(\lambda_2 l), \quad \omega = h_1\eta_1\exp(\lambda_1 l) + h_2\eta_2\exp(\lambda_2 l) \tag{5.9}$$

The constants h_1 and h_2 are found from the initial data $\gamma_0 = \gamma(0)$ and $\omega_0 = \omega(0)$.

If $\Lambda < 0$, the roots (5.8) are complex and solution (5.9) for the case $\Lambda \neq 0$ can be written in terms of real functions as

$$\Lambda > 0,\quad \gamma = \gamma_0\left(\frac{1-u}{1-u_0}\right)\left(\frac{u}{u_0}\right)^{(\nu-1)/2},\quad \omega = \eta\gamma$$
$$\eta = \frac{\eta_1 - u\eta_2}{1-u},\quad u = u_0\exp(-\sqrt{\Lambda}l) \tag{5.10}$$

$$\Lambda < 0,\quad \gamma = \frac{\gamma_0\cos\sigma}{\cos\sigma_0}\exp(\nu(\sigma-\sigma_0)),\quad \omega = \eta\gamma$$
$$\eta = \frac{1}{2}K_\gamma(\sqrt{-\Lambda}\operatorname{tg}\sigma + 1 - \chi),\quad \sigma = -\frac{1}{2}\sqrt{-\Lambda}l + \sigma_0 \tag{5.11}$$

where u_0 and σ_0 are the initial values of u and σ, found from $\eta_0 = \omega_0/\gamma_0$, $\nu = (\chi + 1)/\sqrt{|\Lambda|}$.

When condition (5.7) holds, the real parts of the roots λ_1 and λ_2 are negative, the rest point of system (5.2) in the (γ, ω) plane is a node ($\Lambda > 0$) or a focus ($\Lambda < 0$), and the trivial solution of the system is asymptotically stable. If $K_\omega/K_\gamma > 1$, then $\Lambda > 0$, $\nu < 1$, and $\lambda_1 > 0$. In that case, the rest point is a saddle and the trivial solution of system (5.2) is unstable.

Thus, the motion of the body is stable if condition (5.7) is satisfied. Taking (5.6) into account, we have

$$K_\omega/K_\gamma = 1-(1-A_f/A_m)/(18z_cz_f+1),\quad A_f = -18z_f/P_f \tag{5.12}$$

and condition (5.7) can be rewritten in the form

$$A_m > A_f \tag{5.13}$$

The quantity A_f is determined by the shape of the body and the position of its centre of mass. For a pyramidal body whose mass is uniformly distributed over its volume, $C_m = 3/4$, values of A_f for such bodies are given in Fig. 3 as functions of R_1 ($R_1 = r_1/r_b = \alpha/\beta$). The dashed curves 1–3 are constructed for $\beta = 10°$ for a star-shaped body (curve 1) and for horizontal (curve 2) and vertical (curve 3) rhomboid shapes.

Along the segment of the path where condition (5.5) holds, the characteristics of the medium enter into system (5.2) only via the parameter A_m, and the solution will not depend on the skin friction or

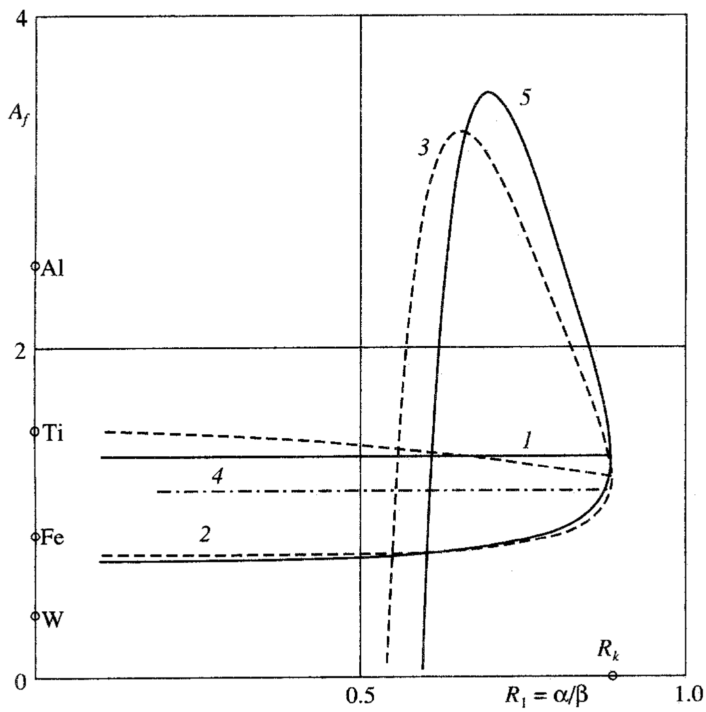

Fig. 3

on the resistant properties of the medium. The value of A_m is independent of the body shape: $A_m = 3A_1/\rho_m \sim \rho_0/\rho_m$; for example, for incompressible elastoplastic media it follows from the first equality of (1.3) that $A_m = 9\rho_0/(2\rho_m)$. Increasing the mass of the body while retaining its shape entails a decrease in the value of A_m, and this, for given U_0, leads to an increase in the length l_m (4.8) of the trajectory of the body. However, if at the same time there is no change in the position of the centre of mass, then the value of A_f remains negligible, and if $A_f > 0$, there is a critical mass $m_f = 3A_1V/A_f$ at which condition (5.13) fails to hold: $A_m = A_f$. This means that the rectilinear motion of bodies of mass $m \geqslant m_f$ is unstable, and when there are small perturbations of the parameters of motion, an investigation of the motion of such bodies may not yield the theoretically predicted depth of penetration.

As an example, Fig. 3 gives values of A_m computed from the first equality of (1.3) for a medium with $\rho_0 = 1500$ kg/m^3. They are indicated by dots on the ordinate axis and correspond to a homogeneous body made of aluminium (Al; $A_m = 2.5$), titanium (Ti; $A_m = 1.5$), steel (Fe; $A_m = 0.85$) or tungsten (W; $A_m = 0.38$). Analysis of Fig. 3 in accordance with condition (5.13) implies, in particular, that for $\beta = 10°$ the motion of star-shaped bodies (see curve 1) constructed for any R_1 and made of steel or tungsten is unstable; whereas the lighter aluminium and titanium presses of the same shape will move stably in the medium.

To analyse the stability of the motion of bodies of whose mass is distributed arbitrarily over the volume, it is convenient to express the stability criterion (5.13) in the form

$$z_y = z_f + A_mP_f/18 > 0; \quad z_y = C_k - C_m, \quad C_k = A_mP_f/18 + 2(1+\alpha^2/P_f)/3 \tag{5.14}$$

where z_y is the stability reserve of the body and C_k is the distance from the vertex of the body to the critical position of the centre of mass at which stability is lost. Hence it follows that, of two bodies with the same shape (same α and P_f) and the same mass distribution over volume (the same C_m), the body of smaller mass (larger A_m) will have the greater stability reserve. This result is in agreement with previous work [8, 9] for slender cones, according to which, other conditions being equal, the motion of cones with relatively lower mass is more stable. The latest analogous result was obtained for slender bodies of revolution [10].

Increasing the base area of a pyramidal body, while keeping its length fixed, leads to an increase in the relative thickness of the body and an increase in the angle β. Under those conditions the body shape is changed, as is the value of z_y. Increasing the angle β for fixed A_m, C_m and R_1 leads to an increase in the values of α and C_k and, consequently, in the stability reserve of the body, z_y.

6. MOTION OF THE CENTRE OF MASS

The trajectory of the centre of mass of the body is determined from Eqs (3.1) and constructed in the fixed system of coordinates $OXYZ$ using the formulae

$$Y(l) = \int_0^l \cos\varphi dl, \quad Z(l) = \int_0^l \sin\varphi dl \tag{6.1}$$

The angle φ is found from the second equation of (4.4) which, when condition (5.5) holds, may be written, taking formulae (4.5) and (5.6) into consideration, in the form

$$d\varphi/dl = A_mP_f\beta(\gamma + \omega z_c) \tag{6.2}$$

The solution of Eq. (6.2) is constructed using solution (5.9) and, using formulae (5.10) and (5.11), we can express it in the form

$$\Lambda > 0, \quad \varphi - \varphi_0 = \zeta_1(u) - u_0\zeta_2(u) \tag{6.3}$$

$$\zeta_j(u) = B_j(1+\eta_jz_c)(1-(u/u_0)^n), \quad j = 1, 2$$

$$B_j = \frac{\gamma_0\beta}{n(1-u_0)\sqrt{\Delta}}, \quad n = \frac{\nu-3}{2} + j$$

$$\Lambda < 0, \quad \varphi - \varphi_0 = \xi(\sigma_0) - \xi(\sigma) \tag{6.4}$$

$$\xi(\sigma) = B_0[h\sin\sigma + \nu\cos\sigma(h - z_c q/(1+\chi))]\exp(\nu(\sigma - \sigma_0))$$

$$B_0 = \frac{\gamma_0\beta K_\gamma\sqrt{-\Delta}}{q\cos\sigma_0}, \quad q = 2\chi(K_\gamma - K_\omega), \quad h = 1 + z_c K_\gamma$$

The rectilinear motion of the body is stable if condition (5.13) is satisfied. Then perturbations in γ and ω will decay with time, and the angle φ will approach an asymptote, which, after the motion has stabilized, will define a new direction of motion of the centre of mass. This direction is determined by the angle $\varphi = \varphi_a$, found from formulae (6.3) and (6.4), which may be written in the form

$$\varphi_a = \varphi_0 + \begin{cases} \zeta_1(0) - u_0\zeta_2(0), & \Lambda > 0 \\ \xi(\sigma_0), & \Lambda < 0 \end{cases}$$

The motion of the body will be defined if D, φ, γ and ω are known as functions of l. If conditions (5.1) and (5.5) are satisfied, they are found from formulae (4.8), (5.10) and (6.3) if $\Lambda > 0$, and from (4.8), (5.1) and (6.4) if $\Lambda < 0$. It follows, in particular, from an analysis of formulae (5.10), (5.11), (6.3) and (6.4) that, along the segment of the path where condition (5.5) holds, the stability of the motion, the angle φ and the shape of the trajectory of the centre of mass (6.1) are independent of the friction and resistant properties of the medium.

As follows from formulae (4.7) and (4.8), $dD/dl > 0$, and as $l \to l_m$ we have $D \to \infty$. Hence a segment $l_d < l_m$ of the path always exists after which, for $l > l_d$, condition (5.5) fails to hold, and then the effect of the parameter D on the solution of system (5.2) cannot be ignored. An increase in the value of D also leads to a decrease in the value of E_1 (see the last formula but one in (4.6)), and it can be shown that there is always a number D_b such that condition (5.4) fails to hold for $D \geqslant D_b$. This means that if criterion (5.13) implies that the equilibrium position of system (5.2) was stable at the starting time, then at $D = D_b$ it will bifurcate (see [14]), and the presence of small perturbations in γ and ω will then cause them to increase. The effect of bifurcation on the solution of the problem of the body's dynamics may be ignored if conditions (5.4) and (5.5) are violated only in a concluding segment of the path which is small compared with the total path length l_m: $(l_m - l_k)/l_m \ll 1$, where $l_k = \min(l_d, l_b) = l(D_b)$. Using formulae (4.8), (5.4) and (5.12), it can be shown that this will be the case if D_0 satisfies both conditions (5.5) and the restriction

$$D_0 \ll D_k, \quad D_k = \frac{P_f(1 - A_f/A_m)}{\alpha^2(1 + 18z_f z_c)} - 1 \tag{6.5}$$

An analytical solution of the problem of the body's dynamics was found above for slender pyramidal bodies (2.3) constructed from pieces of planes tangent to a circular cone with vertex angle 2α. However, if we set $\alpha = \beta$, $P_f = 1$ and $R_1 = 1$ in (4.5) and (4.6), then formulae (4.6) can be used to determine the components of the torque and force experienced by a cone with vertex angle 2β, which is equivalent to pyramidal bodies in respect of the mass, length and base area. Then the solution constructed will be a solution of the problem of the dynamics of the cone, and it will be identical with the solution obtained for star-shaped bodies, apart from the value of z_f. If the position of the centre of mass is the same, the value of z_f and the stability reserve z_y (see (5.14)) for cones is greater, and the value of A_f smaller, than for star-shaped bodies. As an example, curve 4 in Fig. 4 represents the values of A_f for a homogeneous cone with $\beta = 10°$.

A star-shaped body is constructed from pieces of surfaces of two rhomboid bodies and has four planes of symmetry (Fig. 2). The solution of the dynamic problem was obtained for such bodies on the assumption that the motion of the centre of mass takes place in the planes of symmetry that contain the maximum radius of the body. It can be shown that this solution is also true when the centre of mass is moving in the plane containing the minimum radius of the body; this will be confirmed below by numerical calculations.

7. NUMERICAL EVALUATION OF THE ANALYTICAL RESULTS

The analytical results were tested by numerical solution of a Cauchy problem for the system of equations of motion (3.1) and (3.2) of the body, using a fourth-order Runge–Kutta method. When the components

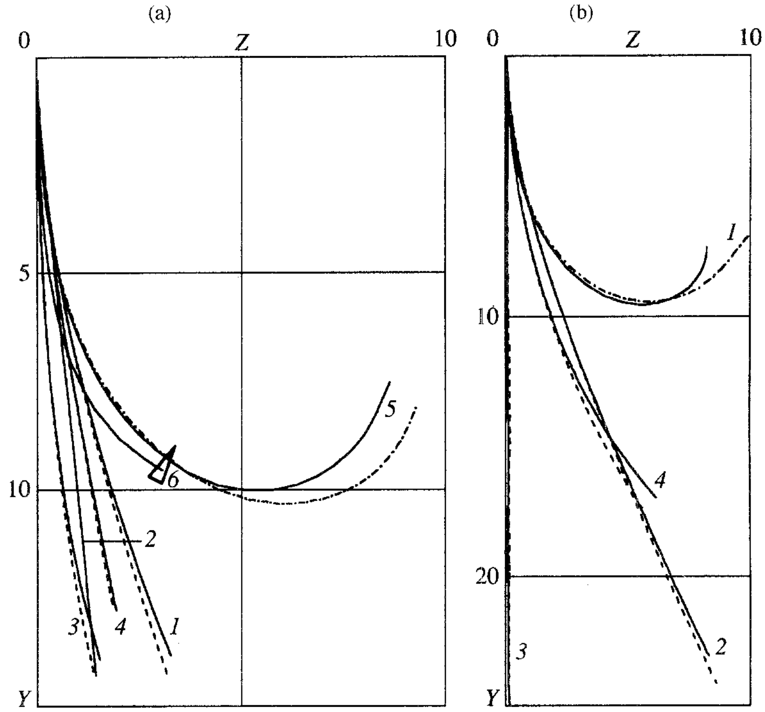

Fig. 4

of the force **F** (1.1) were computed, the surface S (1.5) was defined during the course of the solution, so that it was possible to make allowance for zones of separation of the medium at the body surface. The model medium chosen was one with $\rho_0 = 1500$ kg/m^3, for which the constant parameters of the model (1.2) were computed using formulae (1.3) with $\tau_s = 1$ MPa. It was assumed that $C_1 = 5\tau_s$, which corresponds in order of magnitude to the parameters of a soil of average strength.

The motion of homogeneous titanium, steel and tungsten bodies was considered. The computations were carried out with the following initial data: $U_0 = 600$ m/s, $\varphi_0 = 0$, $\gamma_0 = 0.3$ and $\omega_0 = 0$. The results are represented in Figs 4 and 5 by the solid curves.

Particular attention was devoted to solving the problem of the dynamics of bodies that yield maximum depth of penetration in rectilinear motion. In what follows such bodies will be referred to as optimum. For known parameters of the medium, the value of $\alpha = \alpha^*$ for optimum bodies is determined from the initial velocity U_0 of the body (see [7]). If $U_0 = 600$ m/s we have $\alpha^* = 0.115$, which corresponds to $D_0 = 1.26$. Then $\alpha_1 = 6.6°$, and values of A_f (5.12), computed for homogeneous optimum bodies at $\alpha = \alpha^*$ as functions of R_1, are plotted in Fig. 3 as the solid curves for a star-shaped body (curve 1) and for horizontal (curve 2) and vertical (curve 5) rhomboid shapes.

The trajectories of optimum titanium bodies ($R_1 = 0.66, A_m = 1.5$) are plotted in Fig. 4(a) for a star-shaped body (curve 1) and rhomboid shapes (curves 2 and 3; curve 3 is the trajectory of a vertical rhomboid shape). In that case, according to the stability criterion (5.13), only the vertical configuration is unstable (see Fig. 3), but here increasing the perturbations in γ and ω did not produce a significant deviation of the velocity of the centre of mass from its initial direction, and the final positions of the centre of mass of both rhomboid shapes are almost the same. Curve 4 in Fig. 4(a) also represents the trajectory of a circular cone with $\beta = 10°$, which is equivalent to pyramidal bodies in mass, length and base area. The motion of the cone is stable: $A_f = 1.12$ and $A_m = 1.5$, but the length of its trajectory is less than the length of the trajectories of the optimum bodies by 10%.

Increasing the mass of the bodies keeping the shape the same reduces the value of A_m, and if one considers the same bodies, but made of steel ($R_1 = 0.66$ and $A_m = 0.85$), only the horizontal configuration will be stable. The trajectory of a star-shaped body is then that shown in Fig. 4(a) by curve 5. It is strongly curved, corroborating the theoretical conclusion that simply increasing the mass of the body may not ensure a greater depth of penetration. Further increase in the mass of the body will have the result that, at the very initial stage of the path, the flow of the medium will separate from the body surface, and an increase in the values of the angle of attack and angular velocity of rotation (i.e. in the

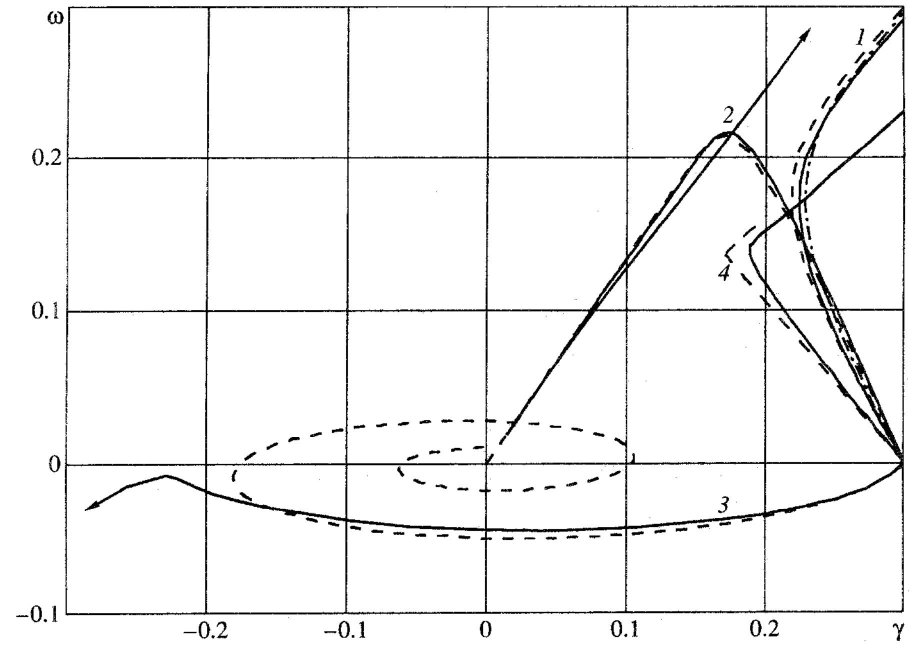

Fig. 5

perturbations of γ and ω) will overturn the body. Curve 6 in Fig. 4(a) represents the trajectory of a star-shaped body made of tungsten ($R_1 = 0.66, A_m = 0.38$), with the body itself shown schematically at the moment of overturn.

The stability of the motion of the body can be improved by increasing its relative thickness. If α is held fixed, this entails a decrease in R_1. The trajectories of optimum steel bodies ($R_1 = 0.5, A_m = 0.85$) are plotted in Fig. 4(b) as curves 1–3 for bodies in the same order as curves 1–3 in Fig. 4(a). Here condition (5.13) fails to hold only for a star-shaped body (see Fig. 3), so that an unstable mode of motion is predicted. Numerical computations confirm this, and the increase in the values of γ and ω for such a body is shown in Fig. 5 by curve 1. The trajectory of a star-shaped body (curve 1 in Fig. 4b) is curved, and its length is significantly less than that of the trajectory of rhomboid bodies, as shown by curves 2 and 3 in Fig. 4(b). The motion of these bodies is stable, the velocity of the centre of mass of the vertical form almost maintaining its initial direction: $\varphi_a = 10^{-2}$. Curves 2 and 3 in Fig. 5 are integral curves in the phase plane (γ, ω) for these bodies. For comparison, curves 4 in Figs 4(b) and 5 plot the trajectory and parameters γ and ω of a steel cone with $\beta = 13°$, which is equivalent to pyramidal bodies in mass, length and base area. The motion of the cone is unstable: $A_f = 0.86$, and its trajectory length is 27% less than those for the optimum shapes.

The results of the computations for the dynamics of star-shaped bodies moving in a plane parallel to the plane of symmetry of the body passing through the minimum radius of the base are plotted as dot-dash curves in Figs 4 and 5. Analysis of the results shows that, until zones of separation of the medium appear at the body surface, the parameters of motion of star-shaped bodies are practically independent of the plane of motion.

The analytical solution of the problem of the body dynamics, as obtained in Sections 4–6, is plotted in Figs 4 and 5 as dashed curves. For an unstable mode of motion, analytical results are given until zones in which the medium separates from the body surface appear, as long as condition (4.1) is maintained. It can be shown that, for a stable mode of motion, the numerical and analytical solutions are practically identical. Differences appear because the numerical solution took all terms occurring in the equations into account, whereas the analytical solution was obtained subject to condition (5.5), which enabled us to ignore the parameter D in solving the problem of the motion of the body about its centre of mass.

The theory predicts the bifurcation of the equilibrium position of system of equations (5.2) as $U_c \to 0$ ($D \to \infty$), which, independently of the original type of solution, causes the values of γ and ω to increase. This has been confirmed by numerical computations, as shown by the arrows attached to curves 2 and 3 in Fig. 5, which originally corresponded to a stable mode of solution. The largest differences between the numerical and analytical solutions of system (3.2) were obtained for a steel vertical shape (see curves

3 in Fig. 5). In that case $D_0\alpha^2/P_f = 0.32$, and the influence of the parameter D on the numerical solution of system (3.2) was felt at the earliest stage of the motion. However, since $D_k = 155$, condition (6.5) was satisfied, and in the final analysis allowance for the parameter D and bifurcation had almost no effect on the solution of the problem of the motion of the body's centre of mass, whose trajectory is plotted as curve 3 in Fig. 4(b).

Note also that the numerical solution was constructed taking into account all forces acting on the body, including the gravity force, whose influence was neglected in the analytical solution. As follows from formulae (4.4) and (4.5), this may be done if

$$A_m U_c^2 \alpha^2 (1 + D) \gg gL$$

which imposes a further restriction on the value of the parameter D

$$D/(1 + D) \ll 3(C_1 + C_2/\alpha)/(\rho_m g L)$$

For media of average strength, $C_1 \sim C_2 \sim 1$ MPa, and when $L \leqslant 10$ m neglect of the gravity force was justified for such media for any values of D.

Thus, the comparison of the results of the numerical and analytical solutions of the problem of the dynamics of the body has shown that the analytical solution, constructed in Sections 4–6 for slender pyramidal bodies under simplifying assumptions, yields a good approximation to the exact solution of the problem, provided conditions (5.1), (5.5) and (6.5) are satisfied at the starting time. The numerical solution yields all types of motion observed in experiments on the high-velocity penetration of solid bodies into dense media [15]: rectilinear, curved, and motion of the body along a trajectory with overturn. It has been shown that the analytical solution may be used to determine the characteristics of the motion of bodies for unstable modes of motion also, but only so long as no zones of separation of the medium have appeared at the body surface.

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